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Gravity wave damping of hydrostatic oscillations for a buoyant disk

By F. W. G. WARREN
Department of Mathematics, Imperial College, London

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A disk (i.e. a body whose maximum thickness is small compared with its lateral dimensions) floats with its central plane of symmetry upright. Its hydrostatic oscillations are lightly damped by the reaction of the gravity waves generated. A damping coefficient is obtained. It is shown that superimposed upon these oscillations is a small displacement which decays with the time t like t^{-4} or t^{-5}.

1. Introduction

When a buoyant vessel is initially disturbed from its position of hydrostatic equilibrium the ensuing motion is damped—among other things—by the generation of surface gravity waves. The general problem has been discussed by several authors. For example, Wehausen (1960, p. 619) outlines a general theory which leads to a non-linear integro-differential equation. He reproduces in graphical form Sretenskii's (1937) numerical calculations for a particular case. Ursell (1964) describes a method which uses a force coefficient $\Lambda(\omega)$ which is obtained by considering the problem of forced oscillations of frequency ω. He discusses the particular case of a half-immersed circular cylinder for which certain properties of $\Lambda(\omega)$ are known or may be found from previous work. He found that the displacement (from the equilibrium position) decays like t^{-2} or t^{-3}, the oscillatory components being exponentially small.

It is worthwhile to note a theory for a class of bodies whose thicknesses are small compared with their lateral dimensions and which float upright, i.e. with their central planes of symmetry vertical. Viscosity is more important here than for fatter bodies, while the wave damping is much lighter; but nevertheless the theory is of interest.

2. Formulation

Consider the disk floating in an upright position at the horizontal interface of two superposed homogeneous liquids which extend to infinity in all directions. When undisturbed the interface defines the (x, y)-plane, with the z-axis positive upwards. Let the central plane of symmetry of the disk coincide with the fixed vertical plane of the y- and z-axes and let the equation of the lateral surfaces of the disk at time t be

$$x = \pm a\xi(y, z, t) = \pm a\xi(y, z - z_0(t)), \tag{2.1}$$

where $2a$ is the maximum thickness of the disk. $z_0(t)$ is the vertical displacement from the position of hydrostatic equilibrium. If $\bar{\rho}$ is the mean density of the upper and lower fluids, then the current density ρ may be expressed as a function of the height z in the form

$$\rho = \rho(z) = (1-\Delta\,\mathrm{sgn}\,z)\bar{\rho}, \tag{2.2}$$

where $0 \leqslant \Delta \leqslant 1$. Then the density difference between the two fluids is $2\Delta\bar{\rho}$. In the position of equilibrium let the area of the horizontal cross-section of the disk at $z = 0$ be

$$2aB, \tag{2.3}$$

so that B is a measure of the width. A mean height, say A, is defined in terms of the mass M_0 of the disk by the relation

$$M_0 = 2aAB\bar{\rho}. \tag{2.4}$$

Since the thickness of the disk is small compared with its lateral dimensions, $a \ll A$, $a \ll B$, and a small thickness parameter is defined by

$$\alpha = 2a/A. \tag{2.5}$$

We wish to solve an initial value problem for the free oscillations of the body. However, it is simpler analytically to consider the equivalent problem in which the velocity and acceleration are zero at $t = 0$ and where a vertical disturbing force equal to

$$M_0 F(t) \tag{2.6}$$

is applied. Initially $(0 \leqslant t \leqslant \delta t)$ the force is just sufficient to hold the disk at rest at a height d_0 above its equilibrium position, so that

$$z_0(0) = d_0 \tag{2.7}$$

and

$$\dot{z}_0(0) = \ddot{z}_0(0) = 0, \tag{2.8}$$

where a dot denotes differentiation with respect to the time. At time $t = \delta t$, $F(t)$ becomes momentarily infinite and thereafter $(t > \delta t)$ vanishes. In this way a vertical velocity U_0 is imparted to the disk just before its release, and the quasi-initial conditions

$$z_0(\delta t) = d_0, \quad \dot{z}_0(\delta t) = U_0 \tag{2.9}$$

hold, where $\delta t \to +0$. This approach simplifies the calculation of the drag made in the next section.

To ensure that the disk's motion is purely vertical, its shape is also symmetric with respect to the z-axis, so that

$$\xi(y,z,t) = \xi(-y,z,t). \tag{2.10}$$

At points of the (y,z)-plane outside the body, $\xi(y,z,t)$ is defined to be zero. When the upwards displacement is z_0 and the interface is undisturbed the increase in the (hydrostatic) thrust is

$$-2aBz_0 2g\Delta\bar{\rho}b(z_0).$$

Here $b(z_0)$ is a factor which corrects for the change with the displacement of the cross-sectional area at $z = 0$. This is $2aB$ only when the displacement is zero (unless the section is locally constant). Thus when z_0 is small (e.g. compared with A) we may write

$$b(z_0) = 1 + b_1 z_0 + O(z_0^2). \tag{2.11}$$

In the general case of oscillations with waves there will be an additional hydrodynamic thrust $\mathscr{R}_d$ which may be written conveniently as

$$\mathscr{R}_d = -\alpha M_0 R(t) = -4a^2 B\bar{\rho} R(t). \tag{2.12}$$

The basic equation of motion of the disk may now be written

$$\ddot{z}_0(t) + \alpha R(t) + n_0^2 b(z_0) z_0(t) = F(t). \tag{2.13}$$

Here n_0 is the hydrostatic frequency:

$$n_0 = (2s^2/A)^{\frac{1}{2}}, \tag{2.14}$$

where

$$s^2 = \Delta g. \tag{2.15}$$

3. The hydrodynamic resistance

In order to estimate the drag we assume that $\alpha \ll 1$. No further assumption is made at this stage (except that the velocity is suitably bounded so as to avoid splashing and/or cavitation). Suppose that $\phi(\mathbf{r}, t)$ is the velocity potential at a current point $\mathbf{r}$ (we will not distinguish between the potentials of the upper and lower fluids). Then the boundary condition at the surface of the disk is given approximately by

$$\left.\frac{\partial \phi}{\partial x}\right|_{x=\pm 0} = \pm a \frac{\partial \xi}{\partial t}, \tag{3.1}$$

if terms $O(\alpha^2)$ are ignored ('thin body' approximation). Physically this equation represents a source distribution on the (y, z)-plane. Since the motion is symmetric with respect to this plane,

$$\phi(x, y, z, t) = \phi(-x, y, z, t). \tag{3.2}$$

Again for small α, the wave amplitude is small, and the kinematic condition at the interface implies that $\partial\phi/\partial z$ is approximately continuous there ('infinitesimal wave' approximation). The strength of the vortex sheet at this surface is measured by G defined as

$$G(x, y, t) = \tfrac{1}{2}(\phi_{z=+0} - \phi_{z=-0}). \tag{3.3}$$

Similarly, continuity in pressure at the interface is expressed by continuity of

$$\rho\left(\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z}\right)$$

at $z = 0$, approximately, if α is sufficiently small. (Lamb 1932, p. 375). In terms of G, this condition may be expressed as

$$\frac{\partial^2 G}{\partial t^2} = s^2\left(\frac{\partial \phi}{\partial z}\right)_{z=0} + \tfrac{1}{2}\Delta \frac{\partial^2}{\partial t^2}(\phi_{z=+0} + \phi_{z=-0}), \tag{3.4}$$

where s^2 is given by (2.15). The boundary condition at infinity is $\phi_\infty = 0$ and the initial conditions at $t = 0$ are that both ϕ and G and their first derivatives with respect to time vanish; i.e.

$$\phi = \frac{\partial \phi}{\partial t} = G = \frac{\partial G}{\partial t} = 0 \tag{3.5}$$

at $t = 0$ (cf. (2.8).)

Now with regard to the excitation of the fluid caused by the motion of the disk and by the vortex sheet at the interface, the field equation for ϕ reads

$$\nabla^2\phi = 2a\frac{\partial\xi}{\partial t}\delta(x) + 2G\delta'(z),$$

where the prime denotes differentiation with respect to z. Equations (3.1) and (3.3) have been used here. Under a Fourier transform

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, e^{-i(kx+ly+mz)}$$

this equation becomes

$$(k^2+l^2+m^2)\phi(k,l,m,t) + 2a\frac{\partial\xi(l,m,t)}{\partial t} + 2imG(k,l,t) = 0. \tag{3.6}$$

The variables here signify transformed quantities. For example

$$\begin{aligned}\xi(l,m,t) &= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \xi(y,z,t)\, e^{-i(ly+mz)} \\ &= e^{-imz_0(t)}\xi_0\,(l,m),\end{aligned} \tag{3.7}$$

where $\xi_0(l,m) = \xi(l,m,0)$.

From (2.3) it follows that

$$\int_{-\infty}^{\infty} dy \xi(y,z,t)\Big|_{\substack{t=0\\ z=0}} = B, \tag{3.8}$$

while from (2.10)

$$\xi_0(l,m) = \xi_0(-l,m). \tag{3.9}$$

Also it is useful to introduce plane polar co-ordinates in the horizontal (k,l)-plane of wave-number space:

$$k = \kappa\cos\theta, \quad l = \kappa\sin\theta, \quad \kappa \geqslant 0. \tag{3.10}$$

Then from (3.6) we find from an application of the inverse operator

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dm\, e^{imz}$$

that

$$\phi(k,l,z,t) = G(k,l,t)\,e^{-\kappa|z|}\operatorname{sgn} z - \frac{a}{\pi}\int_{-\infty}^{\infty} dm\, e^{imz}\frac{\partial\xi(l,m,t)}{\partial t}\Big/(\kappa^2+m^2). \tag{3.11}$$

A vorticity equation now follows from this and equation (3.4), viz.

$$\frac{\partial^2 G}{\partial t^2} + \nu^2 G = -\frac{a}{\pi}\int_{-\infty}^{\infty} dm\left(is^2 m\frac{\partial\xi}{\partial t} + \Delta\frac{\partial^3\xi}{\partial t^3}\right)\Big/(\kappa^2+m^2), \tag{3.12}$$

where

$$\nu^2 = s^2\kappa = \Delta g\kappa = \tfrac{1}{2}An_0^2\kappa \quad (\nu \geqslant 0). \tag{3.13}$$

Equation (3.12) may be solved for G by a quadrature which involves the kernel $\sin\nu(t-\tau)$:

$$G(k,l,t) = -\frac{a}{\pi}\int_{-\infty}^{\infty} dm \int_0^t d\tau\left(is^2 m\frac{\partial\xi(l,m,\tau)}{\partial\tau} + \Delta\frac{\partial^3\xi(l,m,\tau)}{\partial\tau^3}\right)\sin\nu(t-\tau)\Big/\nu(\kappa^2+m^2), \tag{3.14}$$

where the initial conditions (3.5) have been used. If this expression for G is now inserted into (3.11), an integration by parts (twice) and an application of the inverse transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}$$

yields

$$\phi(x,l,z,t) = -\frac{a}{2\pi^2}\int_{-\infty}^{\infty} dk\, e^{ikx}\int_{-\infty}^{\infty} dm\Big\{(e^{imz}+\Delta e^{-\kappa|z|}\operatorname{sgn} z)\frac{\partial\xi(l,m,t)}{\partial t}$$
$$+(s^2/\nu)(im-\Delta\kappa)e^{-\kappa|z|}\operatorname{sgn} z\int_0^t d\tau \sin\nu(t-\tau)\frac{\partial\xi(l,m,\tau)}{\partial\tau}\Big\}\Big/(\kappa^2+m^2). \tag{3.15}$$

Here the initial conditions (3.5) have been used once more.

Consider now the hydrodynamic pressure. This is given approximately by

$$p = -\rho\,\partial\phi/\partial t,$$

if $\alpha \ll 1$, and so the resultant upward hydrodynamic thrust $\mathscr{R}_d$ on the disk at time t is approximately

$$\mathscr{R}_d = -2a\int_{-\infty}^{\infty} dy\int_{-\infty}^{\infty} dz\,\rho(z)\frac{\partial\phi(0,y,z,t)}{\partial t}\frac{\partial\xi(y,z,t)}{\partial z}, \tag{3.16}$$

if terms $O(\alpha^3)$ are neglected. In this expression we substitute

$$(1/4\pi^2 i)\int_{-\infty}^{\infty} d\lambda\int_{-\infty}^{\infty} d\mu\,\mu\xi(-\lambda,-\mu,t)\,e^{-i(\lambda y+\mu z)}$$

for $\partial\xi/\partial z$, and identify

$$\int_{-\infty}^{\infty} dy\, e^{-i\lambda y}\frac{\partial\phi(0,y,z,t)}{\partial t}$$

as a Fourier transform of $\partial\phi/\partial t$. The expression for the thrust may then be written

$$\frac{i\bar{\rho}a}{2\pi^2}\int_{-\infty}^{\infty} d\lambda\int_{-\infty}^{\infty} d\mu\,\mu\xi(-\lambda,-\mu,t)\int_{-\infty}^{\infty} dz\frac{\partial\phi(x=0,\lambda,z,t)}{\partial t}e^{-i\mu z}(1-\Delta\operatorname{sgn} z). \tag{3.17}$$

Here, $\partial\phi/\partial t$ may be found from (3.15) by a differentiation with respect to the time and the substitutions $x = 0$ and $l = \lambda$. Inserting this into (3.17) we see that the z-integration may be performed. For example we have

$$\int_{-\infty}^{\infty} dz\, e^{-i\mu z-\kappa|z|}(\Delta-\operatorname{sgn} z) = 2(i\mu+\Delta\kappa)/(\kappa^2+\mu^2), \text{ etc.}$$

A substitution for $\xi(l,m,t)$ and $\xi(-\lambda,-\mu,t)$ from (3.7) now enables $R(t)$ (cf. (2.12)) to be written in the approximate form

$$R(t) = M_1\{z_0(t)\}\ddot{z}_0(t) - K\{z_0(t)\}[\dot{z}_0(t)]^2$$
$$+s^2\int_0^t d\tau\int_{-\infty}^{\infty} d\kappa\,\kappa f\{\kappa; z_0(t), z_0(\tau)\}\cos\nu(t-\tau)\dot{z}_0(\tau), \tag{3.18}$$

with an error $O(\alpha)$.

Here

$$f\{\kappa; z_0(t), z_0(\tau)\} = (1/8\pi^4 B)\int_{-\pi}^{\pi} d\theta\int_{-\infty}^{\infty} dm\int_{-\infty}^{\infty} d\mu\frac{\mu m(\mu-i\Delta\kappa)(m+i\Delta\kappa)}{(\kappa^2+\mu^2)(\kappa^2+m^2)}$$
$$\times\,\xi_0(l,m)\xi_0(-l,-\mu)\exp[i\{\mu z_0(t)-mz_0(\tau)\}], \tag{3.19}$$

where λ has been replaced by l, and the double integral $\int_{\infty}^{-\infty} dk \int_{\infty}^{-\infty} dl$ has been transformed into its polar co-ordinate form, cf. (3.10). $M_1\{z_0(t)\}$ is a virtual mass coefficient which derives from the first two terms of the integrand on the right-hand side of (3.15). It is given by an integral similar in structure to (3.19), but its precise form is not required here. Similarly for $K\{z_0(t)\}$.

So far no restriction has been placed upon the amplitude or velocity. Thus (3.18) may be used to estimate the wave energy generated at the interface when the disk ascends from low to high levels (i.e. from $z = -\infty$ to $z = +\infty$) with a constant velocity (see, for example, Warren 1961).

4. The equation of motion

The introduction of (3.18) into (2.13) yields

$$[1+\alpha M_1\{z_0(t)\}]\ddot{z}_0(t) - \alpha K\{z_0(t)\}[\dot{z}_0(t)]^2 + \alpha s^2 \int_0^t d\tau \int_0^{\infty} d\kappa\, \kappa f\{\kappa; z_0(t), z_0(\tau)\} \cos \nu(t-\tau)\dot{z}_0(\tau) + n_0^2 b\{z_0(t)\} z_0(t) = F(t). \quad (4.1)$$

This is a non-linear integro-differential equation for the displacement, with an error $O(\alpha^2)$. Also, it holds when $F(t)$ is an impulsive force, although during the time of its action only the first term on the left-hand side is relevant, viz. the inertia term which includes the virtual mass coefficient. Hitherto no restriction has been placed upon the amplitude; however, to find an approximate solution to the initial value problem we now suppose that the amplitude is small (compared with A) and set $z_0(t) = \epsilon A z(t)$, where $\epsilon \ll 1$. Replacing $F(t)$ by $\epsilon A F(t)$ and dividing by ϵA, (4.1) becomes

$$[1+\alpha M_1\{0\}]\ddot{z}(t) + \alpha s^2 \int_0^t d\tau \int_0^{\infty} d\kappa\, \kappa f\{\kappa; 0, 0\} \cos \nu(t-\tau)\dot{z}(\tau) + n_0^2 z(t) + \epsilon n_0^2 A b_1 [z(t)]^2 = F(t), \quad (4.2)$$

if (2.11) is used and terms $O(\alpha^2)$ and $O(\epsilon^2)$ are neglected. Some care is needed here when dividing by ϵ. If the velocity of the disk is $O(\epsilon)$ (rather than $O(1)$), then a factor ϵ appears in all the estimates *and* errors made in the previous section (cf. the thin body and infinitesimal wave approximations) and the error attached to (3.18) is $O(\epsilon\alpha)$. The error in (4.1) is then $O(\epsilon\alpha^2)$, and so division by ϵ is viable. Finally, set $\epsilon = o(\alpha)$, e.g. $\epsilon = \alpha^2$. The non-linear term on the left-hand side of (4.2) which derives from the hydrostatic thrust may then be ignored. Physically this means that the amplitude of the oscillations must be small compared with the (small) thickness of the disk. This condition may be relaxed if $b_1 = 0$, i.e. if the cross-section is locally constant in the neighbourhood of the interface.

A non-dimensional form of (4.2) now follows if we divide by $[1+\alpha M_1\{0\}]$ and make the successive substitutions

$$n_0^2/[1+\alpha M_1\{0\}] = n^2, \quad (4.3)$$

$$\kappa \to 2\kappa/A, \quad m \to 2m/A, \quad t \to t/n,$$

$$f\{2\kappa/A; 0, 0\} \to \tfrac{1}{2}Af(\kappa), \quad (4.4)$$

and

$$F(t) \to n^2 h(t).$$

$n - n_0$ is the change in frequency from a virtual mass effect. Then (4.2) reduces to

$$\ddot{z}(t) + \alpha \int_0^t d\tau \int_0^\infty d\kappa\, \kappa f(\kappa) \cos \kappa^{\frac{1}{2}}(t-\tau)\dot{z}(\tau) + z(t) = h(t), \tag{4.5}$$

if terms $O(\alpha^2)$ are omitted. Next, from (3.19)

$$f(\kappa) = (1/\pi^2 A^3 B) \int_{-\pi}^{\pi} d\theta \left\{ \int_{-\infty}^{\infty} dm \frac{m(m+i\Delta\kappa)}{m^2+\kappa^2} \xi_0(2\kappa \sin\theta/A, 2m/A) \right\}^2, \tag{4.6}$$

where (3.9) has been used. From (3.8) we see that

$$f(0) = 2B/\pi A. \tag{4.7}$$

In (4.6), κ is real and positive. However, $f(\kappa)$ may be continued analytically into the complex κ-plane. For when $\kappa \geqslant 0$

$$\int_{-\infty}^{\infty} dm \frac{m(m+i\Delta\kappa)}{m^2+\kappa^2} \xi_0(l,m) = \int_{-\infty}^{\infty} dm \int_{-\infty}^{\infty} dz \frac{m(m+i\Delta\kappa)}{m^2+\kappa^2} \xi_0(l,z)\, e^{imz},$$

where for convenience we temporarily revert to dimensional variables. With a little manipulation this latter expression reduces to

$$\pi \int_{-\infty}^{\infty} dz \xi_0'(l,z)\, e^{-\kappa|z|} (\operatorname{sgn} z - \Delta), \tag{4.8}$$

where the prime denotes differentiation with respect to z. Now from the definition (2.10), $\xi_0(l,z)$ is an entire function of l which is real when l is real. Hence from (4.6) and the dimensionless form of (4.8), $f(\kappa)$ is an entire function of κ which is real and positive when κ is real and which is $O(\kappa^{-2})$ (in fact $o(\kappa^{-2})$) when κ is real, large and positive. Again, using (4.8), the Fourier transform $\int_{-\infty}^{\infty} dy\, e^{-ily}$, and the identity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, e^{-i\kappa \sin\theta} = J_0(\kappa),$$

we see that $f(\kappa)$ may be written in the (dimensional) form

$$(1/4\pi B) \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2\, \xi_0'(y_1,z_1)\xi_0'(y_2,z_2) \exp\{-\kappa(|z_1|+|z_2|)\}$$
$$\times (\Delta - \operatorname{sgn} z_1)(\Delta - \operatorname{sgn} z_2) J_0\{\kappa(y_1+y_2)\}. \tag{4.9}$$

This indicates the behaviour of $f(\kappa)$ in the complex plane. Thus for ready guidance $f(\kappa)$ may be compared with

$$\kappa^{-\frac{5}{2}}(1+e^{-2A\kappa}) \cos 2B\kappa, \tag{4.10}$$

when $|\kappa|$ is large. Physically from (4.9) we see that $f(\kappa)$ is related to a quasi weighted mean square value of the vertical rate of change of the disk's thickness at any point.

5. Solution of the initial value problem

Referring to the initial conditions (2.7), (2.8) and (2.9), a suitable choice for $h(t)$ enables the initial conditions attached to (4.5) to assume the form

$$z(0) = d, \quad \dot{z}(0) = U, \tag{5.1}$$

where

$$h(t) = 0, \quad t > 0. \tag{5.2}$$

A choice of methods is available for solving (4.5), and it is worthwhile to note that an integration with respect to the time yields an integral equation for the velocity $\dot{z}(t)$:

$$\dot{z}(t) = U - dt - \int_0^t d\tau \left\{(t-\tau) - \alpha \int_0^\infty d\kappa\, \kappa^{\frac{1}{2}} f(\kappa) \sin \kappa^{\frac{1}{2}}(t-\tau)\right\} \dot{z}(\tau). \tag{5.3}$$

This is a Volterra equation of the second kind whose kernel is a function of $(t-\tau)$ and is a well-known type whose solution may be found in a closed form by a Laplace transform (see, for example, Tricomi 1957, p. 23). However, it is simpler to treat (4.5) directly. Subject to (5.1) and (5.2), a Laplace transform

$$\int_0^\infty dt\, e^{i\omega t}, \quad 0 < \operatorname{ph} \omega < \pi, \tag{5.4}$$

reduces (4.5) to

$$D(\omega, \alpha)\zeta(\omega) = U - id\omega\{1 + \alpha H(\omega)\}, \tag{5.5}$$

where $\zeta(\omega)$ is the transform of $z(t)$, and

$$D(\omega, \alpha) = 1 - \omega^2\{1 + \alpha H(\omega)\}. \tag{5.6}$$

Here we have made use of a Faltung theorem, viz. that the transform of

$$\int_0^t d\tau\, \mathscr{F}_1(t-\tau)\mathscr{F}_2(\tau)$$

is equal to the product of the transforms of $\mathscr{F}_1(t)$ and $\mathscr{F}_2(t)$ (see, for example, Tricomi 1957, p. 24). $H(\omega)$ is given by

$$H(\omega) = \int_0^\infty d\kappa\, \kappa f(\kappa)/(\kappa - \omega^2) \tag{5.7}$$

$$= \int_0^\infty d\kappa f(\kappa) + \omega^2 H_0(\omega^2),$$

where

$$H_0(\omega) = \int_0^\infty d\kappa f(\kappa)/(\kappa - \omega^2). \tag{5.8}$$

The κ-integrals here are discontinuous functions of ω when $\operatorname{Im} \omega$ changes sign. Hence we define (5.7) and (5.8) to hold only for $0 < \operatorname{ph} \omega < \pi$, cf. (5.4). Since $f(\kappa)$ is an entire function of κ, the path of integration may be deformed in the complex κ-plane so to avoid the pole at $\kappa = \omega^2$ when ω crosses the real ω-axis. Cauchy's residue theorem then shows that when the κ-path is returned to the real axis,

$H_0(\omega)$ has an analytic continuation given by

$$\left.\begin{aligned} H_0(\omega) &= \int_0^\infty d\kappa f(\kappa)/(\kappa-\omega^2) - 2\pi i f(\omega^2), \text{ if } \pi < \operatorname{ph}\omega \leqslant \tfrac{3}{2}\pi \\ &= \int_0^\infty d\kappa f(\kappa)/(\kappa-\omega^2) + 2\pi i f(\omega^2), \text{ if } 0 > \operatorname{ph}\omega \geqslant -\tfrac{1}{2}\pi \\ &= \text{P.V.}\int_0^\infty d\kappa f(\kappa)/(\kappa-\omega^2) + \pi i f(\omega^2)\operatorname{sgn}\omega, \text{ if } \omega \text{ is real.} \end{aligned}\right\} \tag{5.9}$$

Here the behaviour of $f(\kappa)$ is known, and for large $|\omega|$ the κ-integral is $O(|\omega|^{-2})$, $0 \leqslant \operatorname{ph}\omega \leqslant 2\pi$. When ω is in the neighbourhood of the origin, however, the integral has a logarithmic behaviour, and

$$H_0(\omega) \sim 2f(0)\log\omega. \tag{5.10}$$

The operator inverse to (5.4) is

$$\frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic} d\omega\, e^{-i\omega t},$$

where $c > 0$. From (5.5)

$$z(t) = \operatorname{Re}\frac{1}{\pi}\int_0^\infty d\omega\, e^{-i\omega t}[U + id\omega\{1+\alpha H(\omega)\}]/D(\omega,\alpha). \tag{5.11}$$

It is readily verified that $D(\omega,\alpha)$ has no zeros in the upper half plane, as is to be expected from stability considerations. From (5.9) $H(\omega)$ is regular in the ω-plane if this is cut along the negative imaginary axis. Comparing $D(\omega,\alpha)$ with the function $1-\omega^2$, an application of Rouché's theorem shows that $D(\omega,\alpha)$ has exactly one zero, $\omega = \omega_1$ say, within a large semicircle $|\omega| = r(\alpha)$, $\operatorname{Re}\omega > 0$, where $r(\alpha)\to\infty$ as $\alpha\to 0$. (A first approximation to $\{r(\alpha)\}_{\max}$ yields a term $O(|\log\alpha|^{\frac{1}{2}})$. This follows from an examination of the large modulus zeros of $D(\omega,\alpha)$.) From (5.8), ω_1 is given approximately by

$$\omega_1 = 1 - \tfrac{1}{2}\alpha\,\text{P.V.}\int_0^\infty d\kappa\,\kappa f(\kappa)/(\kappa-1) - \tfrac{1}{2}\alpha\pi i f(1), \tag{5.12}$$

and so lies just below the real axis at a distance $O(\alpha)$ from $\omega = 1$. Hence in (5.11) the integrand is not uniformly bounded on the path of integration as $\alpha\to 0$, and so the path is deformed to lie well below ω_1. Once clear of this point the path returns to the real axis to avoid a possible encounter with the large modulus zeros. For example, a path L would suffice which consists of the semicircle $|\omega - 1| = 1$, $\operatorname{Im}\omega < 0$, and that part of the real axis where $\omega \geqslant 2$. On such a path the integrand behaves satisfactorily, and from (5.11)

$$\begin{aligned} z(t) = {} & \operatorname{Re}\left[\frac{(d+iU\omega)\,e^{-i\omega t}}{1+\frac{1}{2}\omega(1-\omega^2)(d/d\omega)\log H(\omega)}\right]_{\omega=\omega_1} \\ & + \operatorname{Re}\frac{1}{\pi}\int_L d\omega\, e^{-i\omega t}(U - id\omega)/(1-\omega^2) \\ & + \operatorname{Re}\frac{\alpha}{\pi}\int_L d\omega\, e^{-i\omega t}\omega(U\omega - id)H(\omega)/(1-\omega^2)D(\omega,\alpha). \end{aligned} \tag{5.13}$$

The real part of first integral in the right-hand side vanishes identically, while for large t ($\gg 1$) the major contribution to the second integral comes from the path in the neighbourhood of the origin. Setting $\omega \to -i\omega$, this latter integral equals

$$\mathrm{Re}\frac{i\alpha}{\pi}\int_{L'} d\omega\, e^{-\omega t}\frac{\omega(U\omega+d)\left\{\int_0^\infty d\kappa \frac{\kappa f(\kappa)}{\kappa+\omega^2} - 2\pi i\omega^2 f(-\omega^2)\right\}}{(1+\omega^2)\{1+\omega^2(1+\alpha H(-i\omega))\}},$$

where L' is obtained from L by an anticlockwise rotation of $\frac{1}{2}\pi$ about the origin. So for large t we obtain the approximate value of the integral (see, for example, Erdelyi 1956, p. 30), viz.

$$2\alpha f(0)[\Gamma(5)Ut^{-5}(1+O(t^{-2}))+\Gamma(4)dt^{-4}(1+O(t^{-2}))], \tag{5.14}$$

where $f(0)$ is given by (4.7). In terms of dimensional variables, (5.14) equals

$$\frac{M_0 U\Gamma(5)}{2\bar{\rho}\pi g^3\Delta^3 t^5}(1+O(t^{-2}))+\frac{M_0 d\Gamma(4)}{\bar{\rho}\pi A g^2\Delta^2 t^4}(1+O(t^{-2})).$$

Equation (5.13) embodies the principal results of the thin body theory, i.e. that the hydrostatic oscillations are damped by a factor

$$\exp\left\{-\frac{\alpha\pi}{2}f(1)\,t\right\},$$

(cf. (5.12)), and that superimposed upon these oscillations is a small displacement $O(\alpha)$ which decays like t^{-4} or t^{-5}. Also, the frequency and phase of the oscillations are modified by terms $O(\alpha)$ (cf. (4.3)).

The two-dimensional case carries through. The basic modification is the omission of the factor κ in the integrand in the expression for $H(\omega)$, equation (5.7). This has the effect of changing the decay of the non-oscillatory component to the form

$$\frac{\alpha}{t^2}\left(c_1 d + c_2\frac{U}{t}\right) \quad (c_{1,2} = \text{constant}).$$

These are the decay rates obtained by Ursell (1964) for the realistic case of the circular cylinder for which $\alpha = O(1)$. From the viewpoint of the linearization in α, this seems a coincidence. In the case of finite depth, the principal effect is to modify $H(\omega)$ as follows:

$$H(\omega) = \int_0^\infty d\kappa\frac{\kappa f(\kappa)}{\kappa\tanh\kappa h - \omega^2},$$

where the fluids lie between horizontal planes situated at $z = \pm Ah$. This function is again singular at the origin, and suggests that the decay of the non-oscillatory component is then of the form

$$\frac{\alpha}{ht^2}\left(c_1 d + c_2\frac{U}{t}\right)$$

for three dimensions, and

$$\frac{\alpha}{h^{\frac{1}{2}}t}\left(c_1 d + c_2\frac{U}{t}\right)$$

for two dimensions.

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